

Weierstrass's criterion and compact solitary waves

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2007

Abstract

Weierstrass's theory is a standard qualitative tool for single degree of freedom equations, used in classical mechanics and in many textbooks. In this note we show how a simple generalization of this tool makes it possible to identify some differential equations for which compact and even *semi-compact* travelling solitary waves exist. In the framework of continuum mechanics, these differential equations correspond to bulk shear waves for a special class of constitutive laws.

1 Introduction

A *compact wave* is a robust solitary wave with a compact support, outside of which it vanishes identically. A *compacton* is to a compact wave what a soliton is to a solitary wave with an infinite support; that is, a compacton is a compact wave that preserves its shape after colliding with another compact wave. Rosenau and Hyman (1993) introduced these concepts over a decade ago and a substantial number of differential equations supporting compact waves has since been identified and studied.

From the mathematical point of view, the emergence of such solutions is related to the degeneration of the differential equations of motion at certain points, and to the corresponding failure of the uniqueness theorem at these. Indeed, compact waves are a patchwork made pasting together at degenerate points the different possible solutions (unique in between degenerate points), hence patching together non-unique solutions; it follows that they are not analytic solutions, and in this respect they are substantially different from standard soliton solutions.

Compact waves are weak solutions of differential equations, which are continuous – in contrast to shock waves – but have discontinuous derivatives,

similarly to acceleration waves. A simple introduction to compact waves and to compactons can be found in a recent article by Rosenau (2005).

The main objection facing compact waves is that the link between an *adequate* partial differential equation and a *constitutive* law is often tenuous. Indeed, the generic affirmation that compact waves emerge from a balance between higher nonlinearity and nonlinear dispersion remains vague and esoteric if it is not supported by a clear and rigorous mechanical derivation of the *right* equations.

Destrade and Saccomandi (2006^a, 2006^b) recently proposed a general theory of dispersive nonlinear acoustics and showed how it is possible to derive exact equations governing the propagation of compact shear waves in solids with an inherent characteristic length. By applying standard asymptotic procedures to these exact equations, it is then possible to justify evolutions equations which are similar to the compacton factory known as the $K(m, n)$ *KdV equation*.

From the mechanical point of view, these results were obtained by a careful modelling of the dispersive part of the Cauchy stress tensor. The corresponding constitutive equations unify and explain in depth several theories of weakly *nonlocal* continuum mechanics, such as the α -LANS theory of turbulence, or the Rubin, Rosenau, and Gottlieb (2005) theory of inherent characteristic length.

From the mathematical point of view, the emergence of compact waves is investigated by a simple modification of Weierstrass's theory (a useful qualitative tool for single degree of freedom equations in classical mechanics).

In this note we show how another class of constitutive assumptions can generate the emergence of compact waves and even of *semi-compact waves*, which are travelling solitary waves with a semi-infinite support.

2 A generalization of Weierstrass's discussion

One of the most elegant and powerful tools for the qualitative analysis of one-dimensional Lagrangian conservative motions is *Weierstrass's theory*. When an integral of energy (in a generalized sense) exists, Weierstrass's theory allows us to understand whether the motion of our Lagrangian system is periodic or non-periodic, simply by looking at the roots of the potential function.

Let us imagine that our natural Lagrangian system can be described by a single holonomic parameter q , say. Furthermore, suppose that it is time-independent. The energy integral of such a system is

$$T(q, \dot{q}) - U(q) = E , \quad (1)$$

where T is the kinetic energy, U is minus the potential energy (note the unconventional choice of sign), and the constant E is determined by initial conditions.

Recall that for natural Lagrangian systems, the most general form of the kinetic energy is $T = a(q)\dot{q}^2/2$, where $a = a(q)$ is a positive function of q only; also, $T = 0$ only if $\dot{q} = 0$. We will thus assume (1) is in the form

$$\frac{1}{2} a(q) \dot{q}^2 - U(q) = E. \quad (2)$$

It follows that our motion is confined to those configurations where $U(q) + E \geq 0$. The special configurations \bar{q} (say) where $U(\bar{q}) + E = 0$ and $U'(\bar{q}) \neq 0$ are called *barriers* because they cannot be crossed by the motion of $q(t)$: they split the range of possible values for q into allowed and prohibited intervals. Points with $U(\bar{q}) + E = 0$ and $U'(\bar{q}) = 0$ are *soft barriers* and separate two allowed intervals (see below).

Solving first (2) for \dot{q} , and then differentiating with respect to t , we find in turn

$$\dot{q}^2 = 2 \frac{U(q) + E}{a(q)}, \quad \ddot{q} = \frac{1}{a(q)} U'(q) - \frac{a'(q)}{a^2(q)} [U(q) + E]. \quad (3)$$

Hence at a barrier \bar{q} which is a *simple root* of $U + E = 0$ (that is, such that $U(\bar{q}) + E = 0$ and $U'(\bar{q}) \neq 0$), we have $\ddot{q} \neq 0$. Such a barrier is called an *inversion point*, because the motion reverses its course after reaching it.

We now separate the variables in (3)₁ and integrate to find

$$t = \pm \int \sqrt{\frac{a(q)}{2[U(q) + E]}} dq. \quad (4)$$

At a (soft) barrier which is a double root of $U + E = 0$ (that is, such that $U(\bar{q}) + E = U'(\bar{q}) = 0$), the integral diverges. Thus a soft barrier is also called an *asymptotic point* because it takes an infinite time to reach it.

3 Wave propagation

Now we turn to wave propagation. Consider the case of *semi-linear wave equations* in the unknown field $u = u(x, t)$,

$$u_{tt} - c^2 u_{xx} = F(u), \quad (5)$$

where c is a constant and F a nonlinear function of u . Then for travelling waves of speed v , i.e. for u in the form

$$u(x, t) = \varphi(z), \quad z := x - vt, \quad (6)$$

we obtain the second order differential equation

$$(v^2 - c^2)\varphi'' - F(\varphi) = 0. \quad (7)$$

Multiplying by φ' and integrating, we find an equation of the same form as the first equation in (3), where a is the constant $v^2 - c^2$ and U is the anti-derivative of F ; note that E is now related to the integration constant.

Weierstrass's theory tells us that a periodic wave corresponds to the existence of two consecutive inversion points; that a pulse solitary wave with infinite tails corresponds to the existence of an asymptotic point followed by an inversion point; and that a kink solitary wave with infinite tails corresponds to two consecutive asymptotic points; see Peyrard and Dauxois (2004) or Kichenassamy and Olver (1992) for similar discussions.

Consider the case of the following fully *non-linear wave equations*,

$$u_{tt} - c^2 u_{xx} - c_{\text{NL}}^2 (u_x^3)_x = F(u), \quad (8)$$

where c_{NL} is a constant. The travelling wave reduction (6) yields

$$(v^2 - c^2)\varphi'' - c_{\text{NL}}^2 [(\varphi')^3]' = F(\varphi). \quad (9)$$

Here Weierstrass' discussion must be modified, sometimes to dramatic effect.

Take for instance the degenerate case $v^2 = c^2$. In that case, a first integral of (9) is

$$(\varphi')^4 = 2 \frac{U(\varphi) + E}{a}, \quad (10)$$

where $U(\varphi) \equiv -\int F$, E is a constant of integration, and $a = 3c_{\text{NL}}^2/2$.

We can still conduct an analysis *à la* Weierstrass, but we find that the fourth-order power above introduces some new features, not present in mechanical conservative Lagrangian systems. Indeed, now the barriers $\bar{\varphi}$ are attainable not only when they are simple roots of $U + E = 0$, but also when they are double roots. This is the case because the analogue to (4) is here

$$z = \pm \int \sqrt[4]{\frac{a}{2[U(\varphi) + E]}} d\varphi, \quad (11)$$

and the integral converges for double roots.

Saccomandi (2004) gives a detailed discussion on this possibility and explains its consequences by the failure of the Lipschitz condition, leading to a possible lack of uniqueness, and eventually to the possibility of compact waves; see Destrade and Saccomandi (2006^a, 2006^b) for examples.

In the present note we study yet another possibility for Weierstrass's theory, touched upon by Ferrari and Moscatelli (1997), Rosenau (2000), and

Gaeta, Gramchev and Walcher (2006); namely the case of barriers which are roots of *non-integer order* to the equation $U + E = 0$.

For a first glimpse at what may happen in this case, we consider in turn two examples with roots of fractionary order.

Example 1. In the first example, we take $E = 0$, $a = 2$, and

$$U(q) = \sqrt{q} (1 - \sqrt{q}) . \quad (12)$$

Here we record two barriers: $\bar{q}_1 = 0$ and $\bar{q}_2 = 1$. The integral in (4) yields

$$t = -2 \sqrt{\sqrt{q} - q} + \arcsin(2\sqrt{q} - 1) . \quad (13)$$

Clearly, both barriers can be reached in finite times: with our choice of the integration constant, these are $t = -\pi/2$ for \bar{q}_1 , and $t = 0$ for \bar{q}_2 . From (3) the acceleration is

$$\ddot{q} = \frac{1 - 2\sqrt{q}}{2\sqrt{q}} . \quad (14)$$

At $\bar{q}_2 = 1$ the acceleration is not zero, and the motion reverses; at $\bar{q}_1 = 0$ however, the acceleration blows up!

Example 2. In the second example, we take $E = 0$, $a = 2$, and

$$U(q) = q^{4/3} (1 - q^{1/3}) . \quad (15)$$

Here we record two barriers as well, again $\bar{q}_1 = 0$ and $\bar{q}_2 = 1$. We perform the integral in (4) for $t \in [-3\sqrt{2}, 3\sqrt{2}]$ and solve it explicitly for q as

$$q(t) = \left(1 - \frac{t^2}{18}\right)^3 . \quad (16)$$

Clearly again, the barriers are reached in finite times: $\bar{q}_1 = 0$ at $t = \pm 3\sqrt{2}$ and $\bar{q}_2 = 1$ at $t = 0$. We also find that the acceleration is given by

$$\ddot{q} = \frac{1}{3} q^{\frac{1}{3}} (4 - 5q^{\frac{1}{3}}), \quad (17)$$

making it clear that the barrier $\bar{q}_2 = 1$ is a configuration associated with a finite force, whereas the barrier $\bar{q}_1 = 0$ is associated with an equilibrium. Furthermore, note that the right hand-side of (17) does not satisfy the Lipschitz condition at $q = \bar{q}_1 = 0$. This allows non-uniqueness of solution, and in fact we can patch together a compact solitary kink: this is equal to one for $t < 0$, then for $0 < t < 3\sqrt{2}$ it is accelerated to negative velocity and

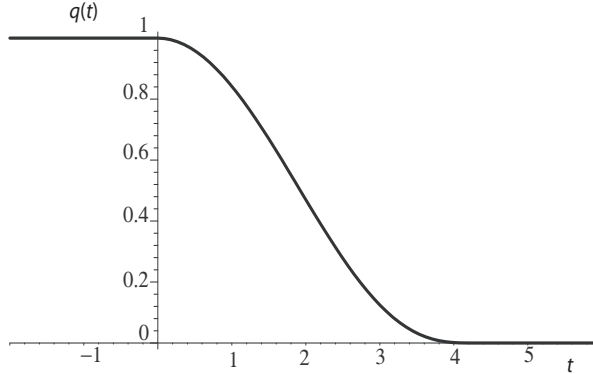


Figure 1: Example 2. Compact solitary kink wave, patched together using fractionary roots in Weierstrass's discussion, with a potential of the form: $U(q) = q^{\frac{4}{3}}(1 - q^{\frac{1}{3}})$.

decreases to reach zero at time $t = 3\sqrt{2}$, and then stays there for $t > 3\sqrt{2}$; see Figure 1.

These two examples highlight the complexity and the richness of the situations arising when considering non-integer barriers.

Let us now turn to a more general (non-fractionary) case. We take $E = 0$, $a = 2$, and

$$U(q) = q^{2r} V(q^2), \quad (18)$$

where V , and thus U , have a barrier $\bar{q} > 0$ ($V(\bar{q}^2) = 0$). Then by (3)₂, the acceleration is

$$\ddot{q} = r q^{2r-1} V(q^2) + q^{2r+1} V'(q^2). \quad (19)$$

We should consider several cases, depending on the value of $r > 0$.

If $2r < 1$, then \ddot{q} blows up at the barrier $q \rightarrow 0$, similarly to the first example above. For this case we conclude that sublinear ($2r < 1$) roots correspond to singular points which cannot be reached in any way.

If $2r = 1$ or $2r = 2$, we recover the already discussed cases of simple and double roots.

If $2r \geq 2$, the integral in (4) diverges and the barrier at zero cannot be reached in a finite time – it is an asymptotic point.

If $1 < 2r < 2$ however, the integral is finite and the barrier zero is reached in a *finite* time. Also, by (19), $\ddot{q} = 0$ at that barrier, and we thus *arrive at an equilibrium configuration with null velocity*, a situation which gives rise to a predicament: will the motion settle on this equilibrium configuration *ad infinitum* or will it reverse its course?

In fact, the Cauchy problem is ill-posed here, and the accompanying lack of uniqueness gives us some latitude to patch together compact waves (similarly to the second example above) or semi-compact waves, as is seen in the next section.

4 Semi-compact shear strain waves

Destrade and Saccomandi (2006^a, 2006^b) show that the motion of transverse strain waves in nonlinear dispersive solids is governed by the following equation,

$$(\mu W)_{xx} + (\alpha W_{tt})_{xx} = \rho W_{tt}, \quad (20)$$

where x is the direction of propagation, W is the transverse strain, and ρ is the mass density. The constitutive parameters are $\mu = \mu(W^2)$, the generalized *shear modulus of nonlinear elasticity*, and α , the *dispersion parameter* of Rosenau et al. (1995). For simplicity here, α is taken constant and the strain wave is linearly polarized, travelling with speed v , see (6). Then $W = W(x - vt)$ and (20) leads to

$$\mu W + \alpha v^2 W'' = \rho v^2 W. \quad (21)$$

Next we write μ in the form

$$\mu(W^2) = \mu_0 - (\alpha \mu_0 / \rho) [r W^{2r-1} V(W^2) + W^{2r+1} V'(W^2)], \quad (22)$$

where μ_0 is the ground state shear modulus, $r > 1/2$, and V is an as yet arbitrary function. For bulk waves, v is arbitrary and here we fix it at the sonic speed $v \equiv \sqrt{\mu_0 / \rho}$. Then (21) is exactly of the same form as (19).

We are thus entitled to consider the possibility of barriers of non-integer order for the wave. For instance, take V in the form

$$V(W^2) = \gamma \frac{\rho}{\alpha} (J_m - W^2)^n, \quad (23)$$

where $\gamma > 0$, $J_m > 0$, and $n > 1$ are constants. Integrating (22) with this choice gives

$$W'^2 = \gamma \frac{\rho}{\alpha} W^{2r} (J_m - W^2)^n. \quad (24)$$

Then the following change of variable and rescaling of function

$$\xi \equiv \left[\gamma \frac{\rho}{\alpha} J_m^{r+n-1} \right]^{\frac{1}{2}} z, \quad \omega(\xi) \equiv W([z(\xi)] / \sqrt{J_m}), \quad (25)$$

give the following non-dimensional version of the governing equation,

$$\dot{\omega}^2 = \omega^{2r} (1 - \omega^2)^n. \quad (26)$$

Now we briefly discuss whether (22)-(23) constitutes a reasonable shear response for a nonlinear solid. We recall that the shear stress τ (say) necessary to maintain a solid in a static state of finite shear with amount of shear K (say) is $\tau = \mu(K^2)K$, given here by

$$\frac{\tau}{\mu_0} = K - \gamma \left[r K^{2r} (J_m - K^2)^n - n K^{2r+2} (J_m - K^2)^{n-1} \right]. \quad (27)$$

First we see that K must not be allowed to go too far beyond $\sqrt{J_m}$ for τ to remain positive. In practice, this means that for a given material we must fix J_m beyond the maximal shear allowed before its rupture. Note that the actual value of J_m has no bearing on the existence and characteristics of the shear wave because it does not appear in (26).

Second we remark that the graph of $\tau(K)$ and the graph of $\mu_0 K$ (corresponding to a material with a linear shear response) cross – independent of γ – for $K = 0$, $K = \sqrt{r J_m / (r + n)}$, and $K = \sqrt{J_m}$. The slopes of the $\tau(K)$ graph at $K = 0$ and at $K = \sqrt{J_m}$ are defined (and equal to μ_0) when

$$2r > 1, \quad \text{and} \quad n > 2. \quad (28)$$

(Of course these slopes are also defined at $2r = 1$ and $n = 2$, but we leave those special cases aside because they lead to simple and double roots in Weierstrass's discussion, already treated above.) The slope of the $\tau(K)$ plot at $K = \sqrt{r J_m / (r + n)}$ is always greater than μ_0 . It follows that between $K = 0$ and $K = \sqrt{r J_m / (r + n)}$, the solid is strain-softening in shear, and that between $K = \sqrt{r J_m / (r + n)}$ and $K = \sqrt{J_m}$, the solid is strain-hardening in shear.

Third we make sure that the shear response is a monotone increasing function of K , by taking γ small enough so that the equation $\tau'(K) = 0$ has no root in the $[0, \sqrt{J_m}]$ interval. Note that the actual value of γ does not affect the non-dimensional equation of motion (26).

Figure 2 displays some examples of shear stress responses satisfying the requirements just evoked.

Having checked that the shear response is sound, we may now look for solutions to the non-dimensional equation (26) describing solitary wave solutions to the original equation (20); these are homoclinic or heteroclinic solution to (26).

Owing to (28), the barrier $\bar{\omega} = 1$ is necessarily an asymptotic point. Taking $r \geq 1$ also creates an asymptotic point barrier $\bar{\omega} = 0$, leading to

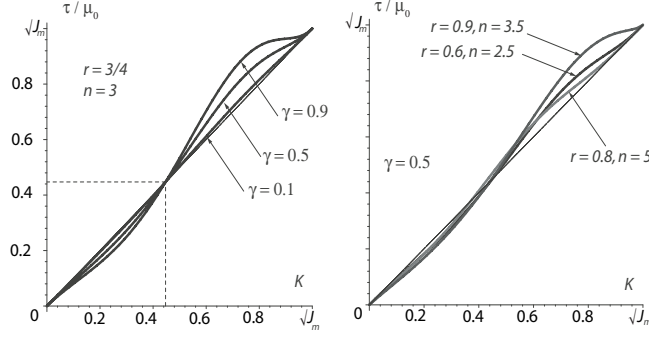


Figure 2: Shear stress response obtained by varying the constitutive parameters in (27). The plots cross the linear stress shear response (thin straight line) at $K = 0$, $K = \sqrt{rJ_m}/(r+n)$ (indicated by dashed lines on left graph), and $K = \sqrt{J_m}$.

a solitary kink with tails of infinite extend. However, taking $1/2 < r < 1$ gives a barrier reachable in a finite time. The result is a *semi-compact wave*, coming from value 1 at $-\infty$ and decreasing to zero, which it reaches in a finite time with zero speed and zero acceleration. It may then remain at this value zero. For Figure 3 we took the case $r = 3/4$, $n = 3$, and chose $\omega(0) = 0$ to fix the value of the integration constant.

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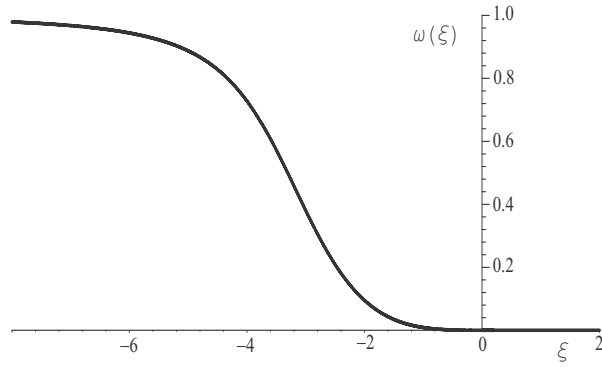


Figure 3: Semi-compact solitary kink wave in a dispersive nonlinear solid with shear stress response given by (27) at $r = 3/4$, $n = 3$.

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